CHANNEL FLOW OF AN ANISOTROPICALLY CONDUCTING MEDIUM IN A ZONE OF ENTRY INTO A MAGNETIC FIELD

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A considerable number of papers are devoted to the problem of the deformation of a plane flow of a conducting liquid moving through a channel $|x| < \infty$, $0 \le y \le h = \text{const}$ in a zone of entry into a magnetic field $B = (0, 0, B_* \eta(x))$, where $\eta(x)$ is the Heaviside unit function ($\eta(x) = 0$ for $x < 0$ and $\eta(x) = 1$ for $x > 0$). Apparently the first paper in this direction was that of Shercliff [1, 2] in which the asymptotic (for $x \to \infty$) profile of a perturbed velocity was determined for a flow of an isotropic conducting liquid in a channel with nonconducting walls. The flow considered by Shercliff takes place in magnetohydrodynamic flowmeters. Complete calculation of the perturbed flow of an isotropic conducting liquid in the channel of a magnetohydrodynamic generator is carried out in [3]. Asymptotic velocity profiles in the channel of a magnetohydrodynamic generator, with ideally segmented electrodes and the flow of an anisotropically conducting medium along them, were found in [4]. General formulas for the calculation of the asymptotic velocity profile, from the known distribution of the perturbing forces along the channel, are presented in [5]. In [6, 7] the Green function is proposed for the solution of the equation for the stream function of the perturbed flow. Finally, in [8], the solution for the perturbed flow of an anisotropically conducting liquid in a channel with continuous electrodes is described by means of the Green function, and the asymptotic profiles of the velocity are calculated.

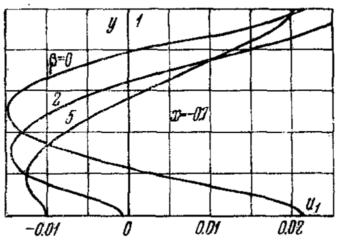

Fig. 1

In this paper the flow of an anisotropically conducting liquid is determined in a channel with ideally segmented electrodes. The solution is set up with the aid of the Fourier method. The resulting series, in which the slowly converging part can be related to the asymptotic profile [4] calculated from the solution of an ordinary differential equation, make it possible to determine the velocity field rapidly. A detailed deformation pattern of the velocity profile is set up. Certain general properties of the flow in a zone of entry into a magnetic field are noted; with the aid of these it is possible to discover the error in the calculations [8].

Let us consider the flow of an incompressible ($\rho = \text{const}$) liquid in a channel $|x| < \infty$, $0 \le y \le 1$ within a transverse magnetic field $b(x) = \eta(x)$. The flow, which is not perturbed by the magnetic field, is assumed to be homogeneous: $\mathbf{v} = (1, 0, 0)$, $p = p_0 = \text{const}$, while the magnetic Reynolds numbers are small. Such a flow is described by the following system of equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial y}{\partial y} = -\frac{\partial p}{\partial x} + Nbj_y, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - Nbj_x \quad \left(N = \frac{\sigma B_*^2 h}{c^2 \rho U}\right), \quad (1)$$

$$j_x = \frac{1}{1+\beta^2 b^2}\left[-\frac{\partial \varphi}{\partial x} + \beta b\left(\frac{\partial \varphi}{\partial y} + ub\right) + vb\right],$$

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0, \quad j_y = \frac{1}{1+\beta^2 b^2}\left[-\frac{\partial \varphi}{\partial y} - \right.$$

$$\left. -\beta b\left(\frac{\partial \varphi}{\partial x} - vb\right) - ub\right], \quad \left(\beta = \frac{e\tau B_*}{mc}\right). \quad (2)$$

In system (1), (2) the longitudinal and transverse velocities u and v, the pressure p, the magnetic field b, the coordinates x and y, the current $\mathbf{j}$ and the potential φ are referred to the velocity U that is averaged across the section, to the velocity head ρU^2, to the characteristic field B_*, to the height h of the channel, and to the quantities $\sigma U B_*/c$ and $U B_* h/c$ respectively. The scalar conductivity σ and the dimensionless Hall parameter β (e and m are the charge and mass of an electron, τ is the average time between electron collisions, c is the velocity of light in vacuum) are assumed to be constant.

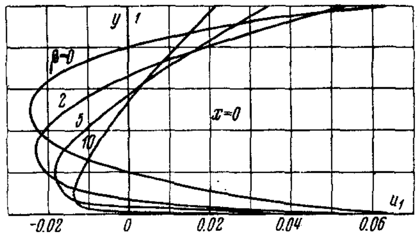

Fig. 2

If the parameter N of magnetohydrodynamic interaction is small, the solution of system (1), (2) by the generally used method can be sought in the form of the following series:

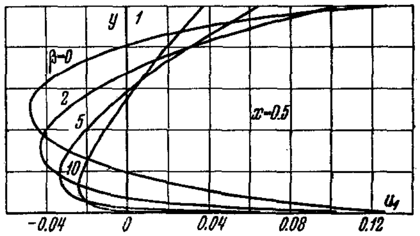

Fig. 3

$$u = 1 + Nu_1 + \ldots, \quad v = Nv_1 + N^2 v_2 + \ldots,$$

$$p = p_0 + Np_1 + \ldots,$$

$$\mathbf{j} = \mathbf{j}_0 + N\mathbf{j}_1 + \ldots, \quad \varphi = \varphi_0 + N\varphi_1 + \ldots. \quad (3)$$

The distribution of the electric current $\mathbf{j}_0$ and the potential φ_0 in the zeroth approximation is found from system (2) in which we must put $v = 0$, $u = 1$. The known solutions of such a system are listed in [9,10]. The perturbations (of first order of smallness) of the gasdynamic parameters are determined from the system

$$\frac{\partial u_1}{\partial x}+\frac{\partial p_1}{\partial x}=bj_{y0}, \quad \frac{\partial v_1}{\partial x}+\frac{\partial p_1}{\partial y}=-bj_{x0},$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0, \quad u_1=v_1=p_1=0 \quad (x\to-\infty). \quad (4)$$

For the stream function $\psi(x,y)$, from (4) without any difficulty we find the equation

$$\Delta\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\int\limits_{-\infty}^{x} j_{x0}\frac{db}{dx}dx$$

$$\left(u_1=\frac{\partial\psi}{\partial y}, \quad v_1=-\frac{\partial\psi}{\partial x}\right),$$

$$\psi(x,0)=\psi(x,1)=0,$$

$$|\psi(x,y)|<C=\text{const} \quad (x\to\infty). \quad (5)$$

Equation (5) was given by Shercliff [2].

If $b(x) = \eta(x)$, the integral on the right side of Eq. (5) equals $j_{x0}(0, y)$ for $x > 0$; it is zero for $x < 0$ and

$$\Delta\psi=0 \quad (x<0), \qquad \Delta\psi=j_{x0}(0,y) \quad (x>0). \quad (6)$$

Solution (6) is sought in the form of the series

$$\psi=\sum_{k=0}^{\infty}\psi_k^{\circ}(x)\sin k\pi y \quad (x<0),$$

$$\psi=\sum_{k=1}^{\infty}\psi_k^{\circ\circ}(x)\sin k\pi y \quad (x>0);$$

The coefficients ψ_k° and $\psi_k^{\circ\circ}$, in accordance with (6), satisfy the equations

$$\psi_k^{\circ\prime\prime}-k^2\pi^2\psi_k^{\circ}=0\ (x<0),$$

$$\psi_k^{\circ\circ\prime\prime}-k^2\pi^2\psi_k^{\circ\circ}=f_k\ (x>0),$$

$$\psi_k^{\circ}(0)=\psi_k^{\circ\circ}(0),\ \psi_k^{\circ\prime}(0)=\psi_k^{\circ\circ\prime}(0). \quad (7)$$

$$\left(j_{x0}(0,y)=\sum_{k=1}^{\infty}f_k\sin k\pi y,\right.$$

$$\left.f_k=2\int\limits_0^1 j_{x0}(0,y)\sin k\pi y\,dy\right). \quad (8)$$

The boundary conditions at the point $x = 0$ are obtained from the condition of continuity for the velocities u and v when $x = 0$. At infinity ($|x| \to \infty$) the functions ψ_k° and $\psi_k^{\circ\circ}$ must be finite.

After solving (7) we find

$$x<0$$

$$\psi=-\sum_{k=1}^{\infty}\frac{f_k}{2k^2\pi^2}e^{k\pi x}\sin k\pi y,$$

$$u_1=-\sum_{k=1}^{\infty}\frac{f_k}{2k\pi}e^{k\pi x}\cos k\pi y,$$

$$x>0$$

$$\psi=-\sum_{k=1}^{\infty}\frac{f_k}{2k^2\pi^2}(2-e^{-k\pi x})\sin k\pi y,$$

$$u_1=-\sum_{k=1}^{\infty}\frac{f_k}{2k\pi}(2-e^{-k\pi x})\cos k\pi y. \quad (9)$$

We draw attention to the fact that to calculate the velocity field for $b(x) = \eta(x)$ we must know only the quantity $j_{x0}(0,y)$. This is explained by the circumstance that, owing to the conservation of the electromagnetic force on the left and on the right of the section $x = 0$, the vorticity $\omega = \partial v_1/\partial x - \partial u_1/\partial y$ is zero for $x < 0$ and is maintained along the streamlines (which in the zeroth approximation coincide with the straight lines y = const) for $x > 0$. But the variation of the vortex on the line y = const in the section $x = 0$ equals $-j_{x0}(0,y)$. Thus, the distribution of the vortex over the entire channel (and hence, over the velocity field) become known, if the axial current $j_{x0}(0,y)$ is given. Mathematically this is shown in Eq. (6).

The velocity of flow $u_1(\infty, y) = u_1^+(y)$ for $x \to \infty$, according to (9), is expressed by the formula

$$u_1^+(y)=-\sum_{k=1}^{\infty}\frac{f_k}{k\pi}\cos k\pi y. \quad (10)$$

But from (8) we have

$$\int\limits_0^y j_{x0}(0,y)\,dy=u_1^+(y)+\sum_{k=1}^{\infty}\frac{f_k}{k\pi}. \quad (11)$$

Taking into account the circumstance that the velocity u_1^+, averaged across the section of the channel, is zero, from (11) we obtain

$$u_1^+(y)=\int\limits_0^y j_{x0}(0,y)\,dy-\int\limits_0^1\left(\int\limits_0^y j_{x0}(0,y)\,dy\right)dy. \quad (12)$$

Thus, the asymptotic profile of the velocity is determined by means of elementary integration. The found function $u_1^+(y)$ can then be used to speed the calculations, since it (see [10]) replaces the slowly converging part in formula (9).

We note that according to (12) $u_1^+(0) = u_1^+(1)$, if the current j_{x0} averaged for the section $x = 0$ is zero. This condition is always satisfied, if the walls of the channel for $x < 0$ are nonconducting and $j_{x0} \to 0$ for $x \to -\infty$. Therefore the graph of the asymptotic velocity given in [8], from which it follows that $u_1^+(0) \neq u_1^+(1)$ (although the above-mentioned condition is satisfied), is incorrect.

It can also be shown that for a channel with nonconducting walls the quantity $u_1^+(0) = u_1^+(1) = u_{1w}^+$ equals $q = Q(\sigma B_*^2 h^2 U^2/c^2)^{-1}$, where Q is the joulean dissipation in the channel, calculated from the current distribution in zeroth approximation.

We also note that, as follows from formulas (9), the axial velocity at the section $x = 0$ is smaller by a factor of two than the velocity for $x \to \infty$.

Let us consider the flow of an anisotropic conducting liquid in a channel whose walls are ideally segmented electrodes for $x > 0$, with the condition $j_y = -(1 - K)$ = const satisfied for them, and which are nonconducting for $x < 0$. The quantity $K \leq 1$ represents a load parameter.

The distribution of electric currents in such a channel for $b(x) = \eta(x)$ is found in [11]. The quantity $j_{x0}(0,y)$ is represented by the formula

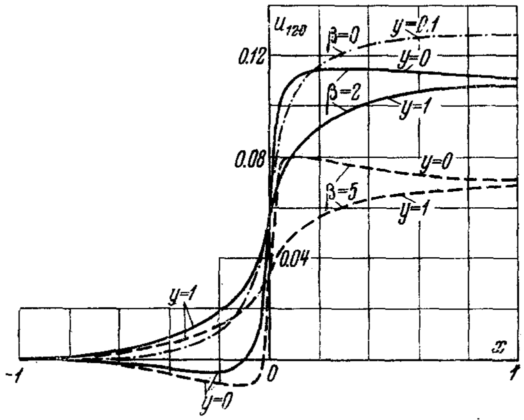
u_{120}
0.12
0.08
0.04
β=0
β=2
β=5
y=0.1
y=0
y=1
y=0
y=1
y=1
y=0
x
-1
0
1

Fig. 4

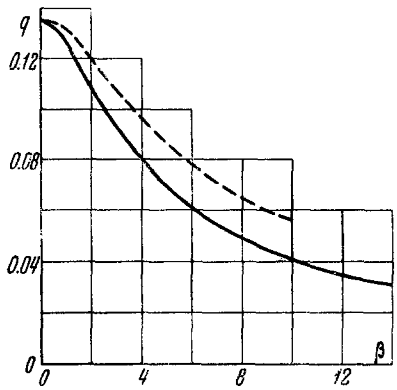
q
0.12
0.08
0.04
0
0
4
8
12
β

Fig. 5

$$j_{x0}(0, y)=\frac{K}{\beta}\left[1-\frac{2}{(4+\beta^2)^{1/2}}\left(\operatorname{ctg}\frac{\pi y}{2}\right)^{1-2\nu}\right],$$

$$\nu=\frac{1}{\pi}\operatorname{arc\,tg}\frac{2}{\beta}\quad(0<\nu<0.5),$$

$$j_{x0}(0, y)=\frac{K}{\pi}\ln\operatorname{tg}\frac{\pi y}{2}\quad(\beta=0). \qquad (13)$$

For K = 0 the solution corresponds to the conditions of a short circuit; for K = 1 it corresponds to the conditions of idling, where the distribution of electric current becomes the same as in a channel with nonconducting walls.

The Fourier coefficients (8) corresponding to this current distribution were found by numerical integration.

We present certain results of the calculation for the case K = 1. The velocity profiles at the section $x = -0.1$ for various β are shown in Fig. 1. For $\beta \neq 0$ the flow ceases to be symmetric about the axis $y = 1/2$. The derivatives $\partial u_1/\partial y$ on the walls are zero. This is connected with the fact that the vorticity ω is zero for $x < 0$, while on the walls $\omega = -\partial u_1/\partial y$.

The profiles $u_1(0, y)$ and $u_1(0.5, y)$ are shown, respectively, in Figs. 2 and 3. As was shown earlier, $u_1(0, y) = 0.5u_1^+(y)$, where $u_1^+(y)$ is the asymptotic velocity profile. For $x > 0$ the derivative $\partial u_1/\partial y$ on the lower wall passes to infinity. This is explained by the fact that $\omega(x, 0) = -j_{x0}(0, 0) = +\infty$ for $x > 0$. Since $\omega(x, 1) = -j_{x0}(0, 1) \neq \infty$ for $\beta \neq 0$, the derivative $\partial u_1/\partial y$ on the upper wall is finite. As the Hall parameter increases, the velocity perturbations decrease in absolute value. Figure 4 shows the velocity distribution along the upper and lower walls. For $x \approx 1$ the quantity u_1 hardly differs from its asymptotic value u_1^+. We note, however, that as the Hall parameter β increases, the convergence to asymptotic values becomes slower.

Since on the walls $j_y = 0$ (for K = 1), in accordance with the first equation in (4), we have $p_1 = -u_1$ for $y = 0$ and $y = 1$. Thus the pressure losses in the channel equal $p_1^{(\infty)} = -u_1^+(0) = -u_1w^+$. The quantity u_1w^+, as was shown above, equals the dimensionless joulean dissipation q. The relationship $q(\beta)$ is shown in Fig. 5. Here the dashed line shows the function $q(\beta)$ obtained in [12], in which, in contrast to [11], the electric field in a channel with nonconducting walls was calculated by the Fourie method. The divergence of the curves is apparently explained by the insufficient accuracy of the calculations in [12]. (In [12] the problem is reduced to the solution of an infinite system of algebraic equations which is replaced by a finite system of the same order for all β. At the same time we know that, as a rule, the convergence of various approximate methods is impaired as β increases.)

If $K \neq 1$, on the walls $p_1 = -u_1$ for $x < 0$ and $p_1 = -u_1 - (1 - K)x$ for $x > 0$. In the region of asymptotic flow $p_1^+ = -u_1^+(0) - (1 - K)x$. The quantity $u_1^+(0)$ in this formula is K times smaller than that given in Fig. 5.

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